Triangulation

Triangulation uses the mathematical techniques of trigonometry, described by the ancient Greeks and preserved through the medieval period by the Moors. The idea is best described by an example. We're going to find the position of St Martin's Church in Dorking using a theodolite and the network of ***triangulation points*** (or ***trig points***) created during the last complete survey of Britain, run by the Ordnance Survey from 1935 to 1962.

A theodolite is a telescope with two scales. One scale shows the angle of the telescope in the horizontal plane and the other in the vertical. The base has adjustable levelling feet so it can be set in the horizontal.

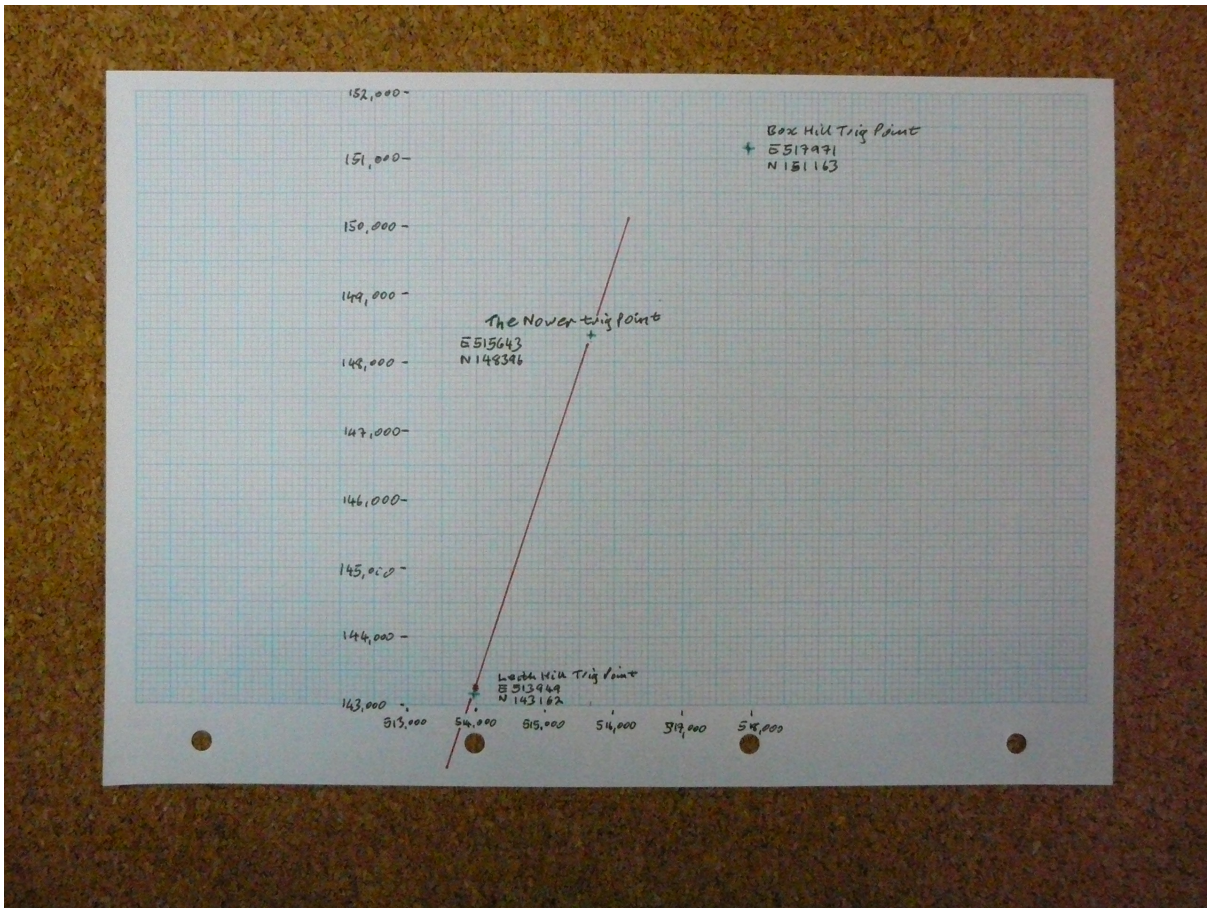
The instrument in the picture is a toy, accurate to maybe half a degree. A real surveyor's theodolite would be accurate to a few seconds of arc, a second being 1/3600 degrees.

Positions on an OS map are specified with respect to an arbitrary point out at sea to the south-west of the UK. Each position is given as the distance to the east of that point in metres (the easting), the distance to the north (the northing) and its height above sea level.

The trigpointing website https://trigpointing.uk/ gives the position of all of the OS trig points.

Visit the website and type Dorking into the search box. That shows you all of the trig points near that town. We're going to use the ones at Leith Hill Tower, The Nower and Box Hill. The search results have a link to a page for each trig point, each giving the position of that trig point. For example, the one at The Nower is at easting 515643 metres, northing 148396 metres.

Mark the positions of the Leith Hill and Nower trig points on graph paper and draw a line joining them. This will be the ***baseline*** of our measurements:



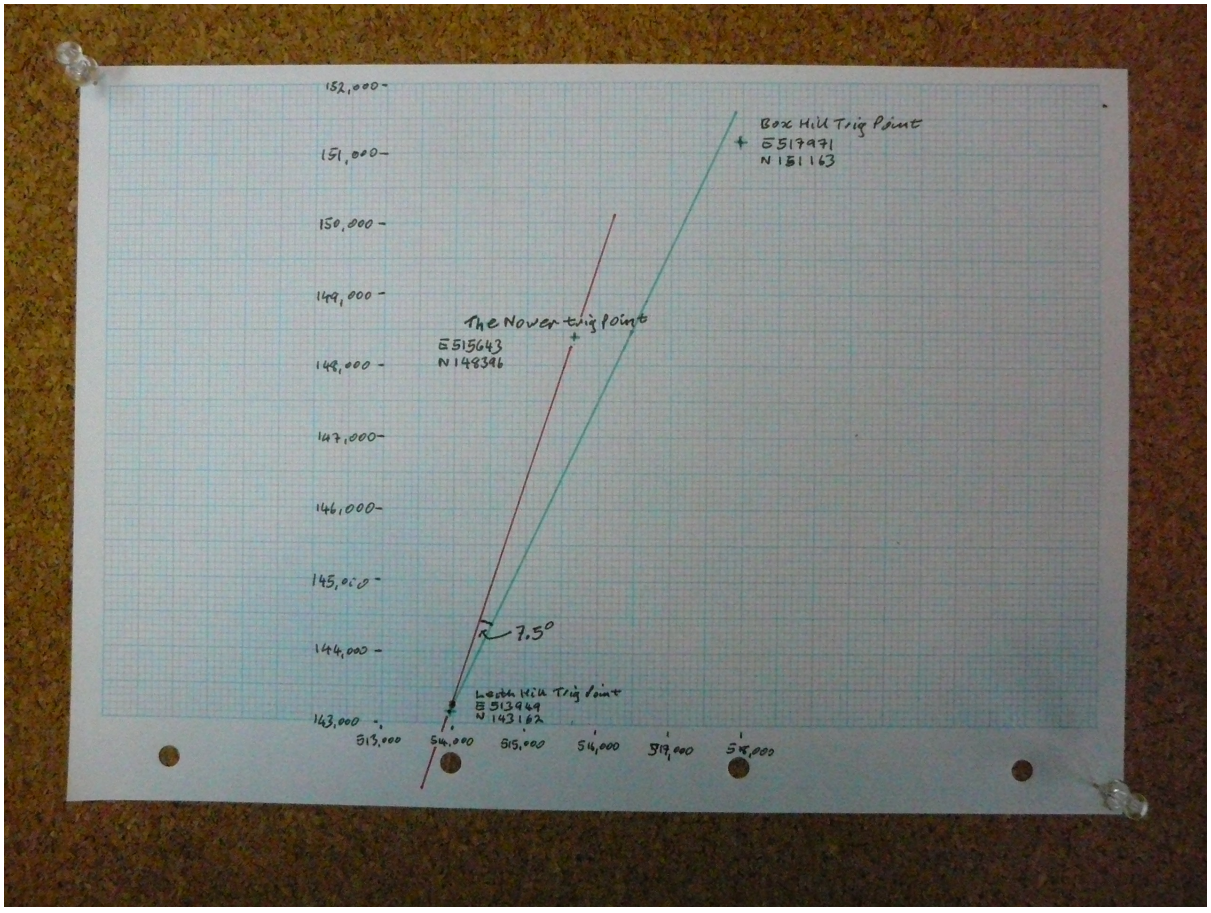
Now we have to take the theodolites to the trig points and take measurements.

The trig point on Leith Hill Tower is marked by a brass bolt in the centre of the roof. There's a parapet around the edge, so a small table and a plumb line will be handy! Place the theodolite above the centre of the bolt.

(OS Surveyors sometimes had to build a 30-foot scaffolding tower to see past various obstacles.)

Turn the theodolite until the zero degree mark on the horizontal scale points directly at The Nower trig point. Adjust the feet to level the instrument. The scale is now aligned with the baseline.

Turn the telescope so that it points to St Martin's Church. Read the angle to the baseline using the horizontal scale – 7.5 degrees. Mark that line on your graph. In my picture the baseline is green:

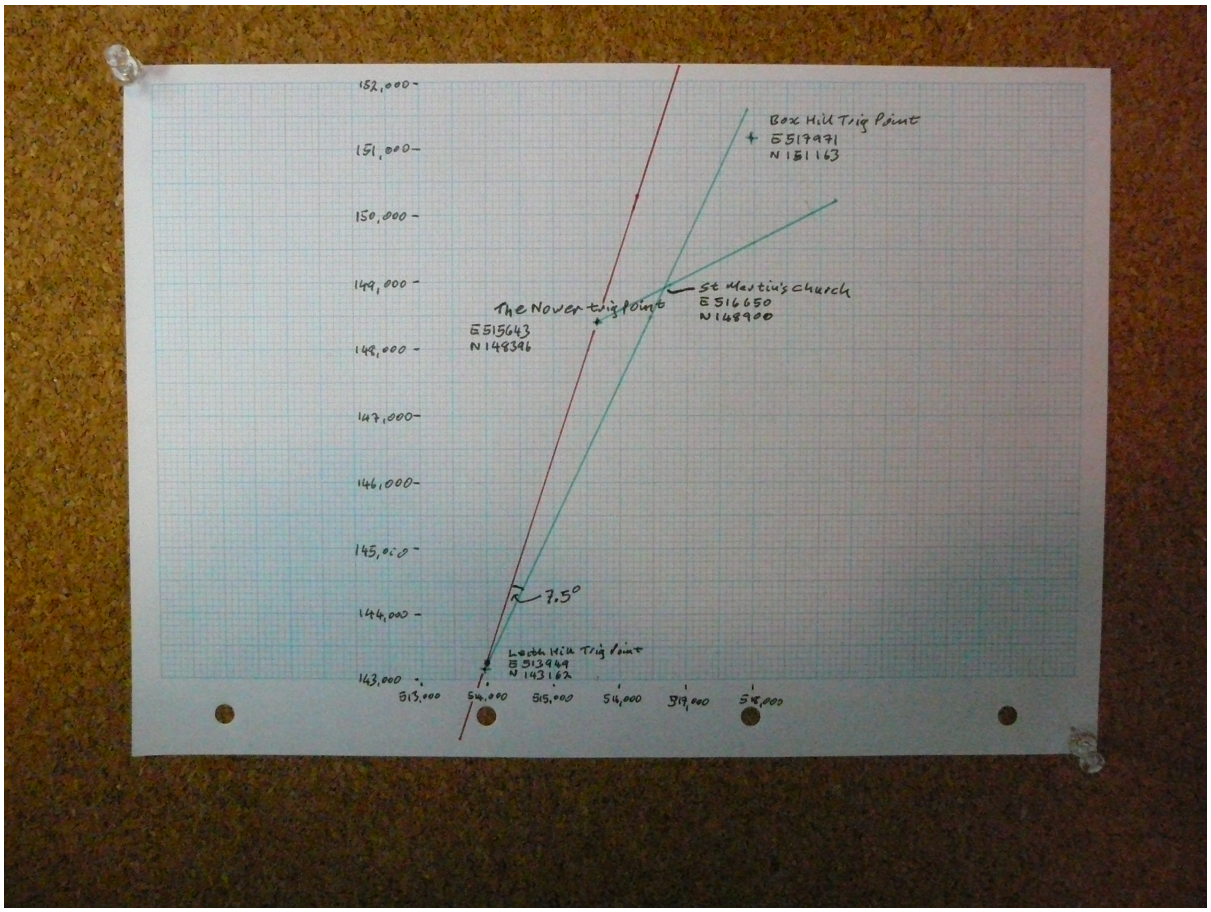


The church is somewhere along that line.

Now go to the trig point at The Nower and put the theodolite on top of it in the centre. Align the instrument to the baseline by pointing the zero degrees mark

at the Leith Hill trig point. Turn the telescope to the church and find the angle between that line and the baseline – 45 degrees.

Now we can draw another green line on the graph, producing a triangle:



The church is at the third corner of the triangle and we can read its position from the graph – Easting 516650 metres, Northing 148900 metres.

The church, the Leith Hill trig point and The Nower trig point are all visible from the Box Hill trig point so we can check our result by making similar observations but with different baselines: Box Hill to The Nower and Box Hill to Leith Hill.

Now we know the position of the church, but not the height above sea level. The theodolite also has a vertical scale which gives you the angle of elevation of the object you are sighting on. The Trigpointing website gives the height above sea level of the trig points. We can use the triangulation technique to find the height above sea level of the church, but that involves drawing triangles in three dimensions.

If you use a real theodolite rather than my toy, your angle measurements will be far more accurate than your picture on graph paper. You can use trigonometry to get a decent result.

Some Basic Mathematics

Let's recall some of the geometry we learned at school.

The angles in a triangle add up to 180 degrees.

The longest side of a right angled triangle is the hypotenuse. Each of the other angles has a side opposite to it and a side adjacent to it.

The sine of the angle is (length of opposite) / (length of hypotenuse).

The cosine of the angle is (length of adjacent) / (length of hypotenuse).

The tangent of the angle is (length of opposite) / (length of adjacent).

Pythagoras' theorem says:

square of (length of hypotenuse) = square of (length of adjacent) +
square of (length of opposite)

We can do all the calculations with a spreadsheet app such as Microsoft Excel. It provides the sine, cosine and tangent functions, plus the reverse functions arcsine, arccosine and arctangent. They work in radians, not degrees, so you need to convert. For example, to get the sine of 30 degrees, you need this spreadsheet formula:

=SIN(30*PI()/180)

The result is 0.5. This formula uses the arcsine function to give back the angle :

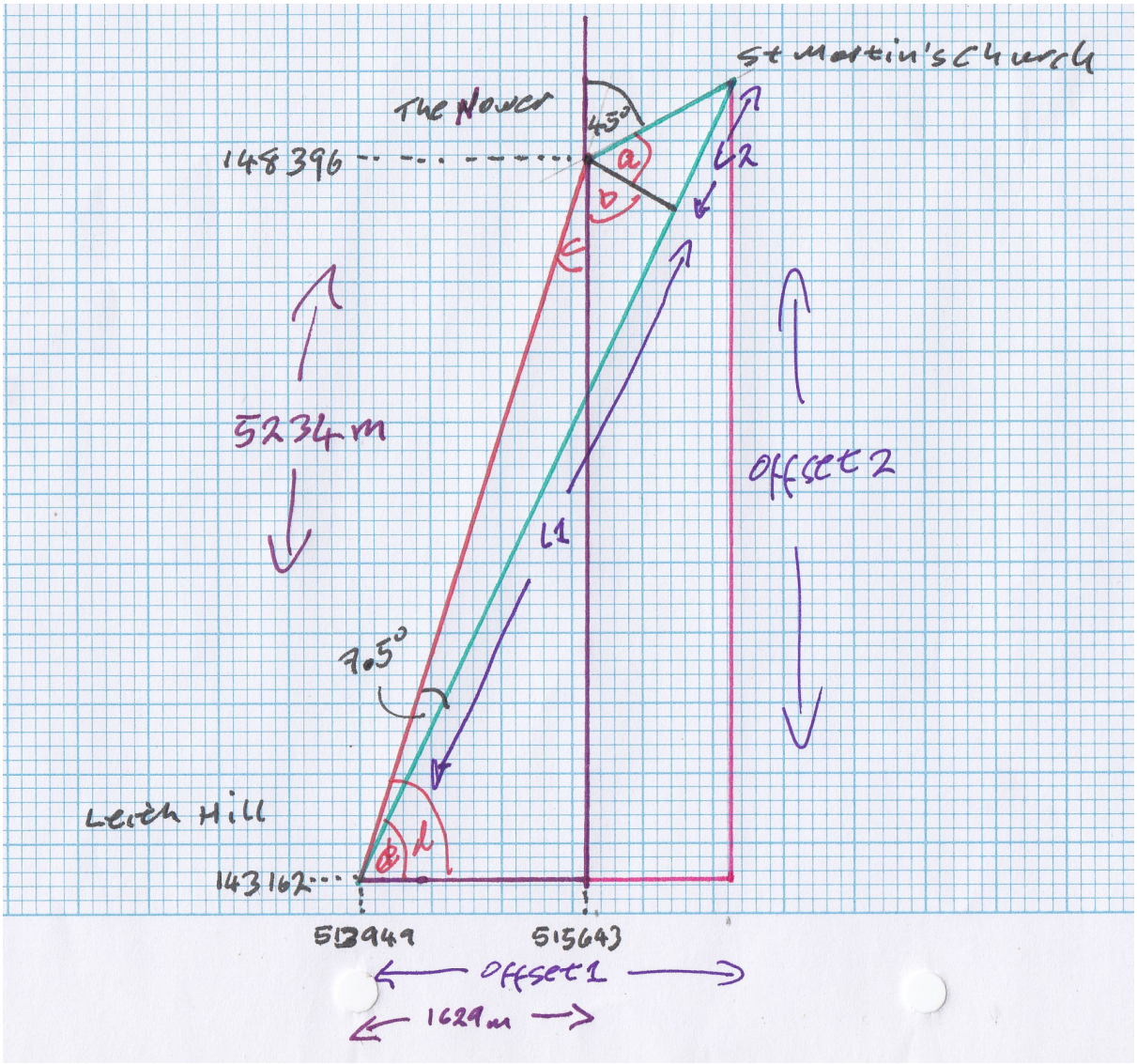
=ASIN(0.5) * (180/PI())

Those tools are enough to calculate the position of the church from the measurements I made. Try it.

My solution is on the next page.

The Calculation

The goal is to find the lengths of the horizontal and vertical red lines in the picture below. These are the offsets of the position of the church from the position of the trig point at Leith Hill. I've labelled them offset1 and offset2.



If I can find the angle e and the lengths l1 and l2 then I can use sines and cosines to find the offsets.

The angles marked a, b, c, d and e are unknown.

We know the eastings and northings of the trigs at Leith Hill and The Nower, which gives the lengths of the two purple lines – 1629 metres and 5234 metres.

Finding e

The two purple lines and the diagonal red line form a right-angled triangle. We can use Pythagoras' theorem to find the length of the red line. I make it 5481.641816 metres.

We can see some things about the angles immediately:

a, b and the 45 degree angle form a straight line,

so a + b + c + 45 degrees = 180 degrees

so a + b + c = 135 degrees

The angles in a triangle add up to 180 degrees

so b + c + 7.5 + 90 = 180 degrees

so b + c = 82.5 degrees.

So a is 52.5 degrees

By similar reasoning

d + c = 90 degrees

We can see that e + 7.5 = d

so e + c = 82.5

The tangent of c is 1629/5234 = 0.311234218

so c = 18.13362713 degrees (using the arctangent)

We know that c + b + 7.5 + 90 = 180

so c + b = 82.5 degrees

so b = 64.36637287 degrees.

We know that e + c = 82.5 degrees

so e is 64.36637287 degrees

Finding the lengths l1 and l2

The red line and the two green lines form an irregular triangle, but we can divide it into two right-angled triangles using the black line. That divides the line we are interested in into two parts, of length l1 and l2. We know the length of the red line and the angle 7.5 degrees.

The red line is 5481.641816 metres long

So the length of the black line is
sine(7.5) X 5481.641816 = 715.4978334 metres

and the length of l1 is
cosine(7.5) X 5481.641816 = 5434.74561 metres

The angle a is 52.5 degrees so the length of l2 is

tan(52.5) X 715.4978334 = 932.454937

The length of the long green line is l1 + l2 = 6367.200541

The Solution

The long green line is the hypotenuse of a right-angled triangle. It's 6367.200541 metres long. The angle at the bottom right corner is e, which is 64.36637287.

offset 1 is the length of the adjacent side, cosine(64.36637287) X 6367.200541 which is 2754.54623

offset2 is the length of the opposite side, sine(64.36637287) X 6367.200541 which is 5740.532884

Adding those offsets to the position of the Leith Hill trig gives the position of St Martin's Church:

easting 516703.5462, northing 148902.5329

Sanity Check

The rough position I got by drawing a graph was

easting 516650, northing 148900

So the northing on my graph is pretty accurate but the easting is out by 54 metres, about two millimetres on the paper. My skill with a ruler and protractor seems to have gone downhill since I was at school.